ISRAEL JOURNAL OF MATHEMATICS, Vol. 48, Nos. 2–3, 1984

INSTANCES OF THE CONJECTURE OF CHANG

BY
JEAN-PIERRE LEVINSKI

ABSTRACT

We consider various forms of the Conjecture of Chang. Part A constitutes an introduction. Donder and Koepke have shown that if ρ is a cardinal such that $\rho \geqq \omega_1$, and $(\rho^{++}, \rho^{+}) \twoheadrightarrow (\rho^{+}, \rho)$, then 0^{+} exists. We obtain the same conclusion in Part B starting from some other forms of the transfer hypothesis. As typical corollaries, we get:

THEOREM A. *Assume that there exists cardinals λ, κ, such that $\lambda \geqq \kappa^{+} \geqq \omega_2$ and $(\lambda^{+}, \lambda) \twoheadrightarrow (\kappa^{+}, \kappa)$. Then 0^{+} exists.*

THEOREM B. *Assume that there exists a* singular *cardinal κ such that $(\kappa^{+}, \kappa) \twoheadrightarrow (\omega_1, \omega_0)$. Then 0^{+} exists.*

THEOREM C. *Assume that $(\lambda^{++}, \lambda) \twoheadrightarrow (\kappa^{++}, \kappa)$. Then 0^{+} exists (also if $\kappa = \omega_0$).*

REMARK. Here, as in the paper of Donder and Koepke, "0^{+} exists" is a matter of saying that the hypothesis is strictly stronger than "$L(\mu)$ exists". Of course, the same proof could give a few more sharps over $L(\mu)$, but the interest is in expecting more cardinals, coming from a larger core model.

THEOREM D. *Assume that $(\lambda^{++}, \lambda) \twoheadrightarrow (\kappa^{+}, \kappa)$ and that $\kappa \geqq \omega_1$. Then 0^{+} exists.*

REMARK 2. Theorem B is, as is well-known, false if the hypothesis "κ is singular" is removed, even if we assume that $\kappa \geqq \omega_2$, or that κ is inaccessible. We shall recall this in due place.

COMMENTS. Theorem B and Remark 2 suggest we seek the consistency of the hypothesis of the form: $(\kappa^{+}, \kappa) \twoheadrightarrow (\omega_{n+1}, \omega_n)$, for κ singular and $n \geqq 0$.

The consistency of several statements of this sort — a prototype of which is $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega_0)$ — have been established, starting with an hypothesis slightly stronger than: "there exists a huge cardinal", but much weaker than: "there exists a 2-huge cardinal". These results will be published in a joint paper by M. Magidor, S. Shelah, and the author of the present paper.

Part A

1. Introduction

These notes treat of some model-theoretic properties, which have been introduced and studied by Chang, Rowbottom, and others.

We recall the definitions (see [9]).

Received February 13, 1984 and in revised form April 2, 1984

DEFINITION 1. Let α, β, γ, δ, μ be infinite cardinals. We set: $(\alpha,\beta) \xrightarrow[<\mu]{} (\gamma,\delta)$ iff the following holds: for every language L, such that $|L| < \mu$, for every distinguished unary predicate $A \in L$, for every L-structure $M = (M, A^M, L^M)$, such that $|M| = \alpha$ and $|A^M| = \beta$, there exists $N = (N, A^M \cap N, L^M \restriction N)$, such that:

(1) $N \lhd M$ (which means: "is elementary substructure of ..."),

(2) $|N| = \gamma$ and $|A^M \cap N| = \delta$.

When $\mu = \omega_1$, we just write: $(\alpha,\beta) \twoheadrightarrow (\gamma,\delta)$. When $\mu = \nu^+$, we write: $(\alpha,\beta) \xrightarrow[\leq\nu]{} (\gamma,\delta)$.

The definition is relevant only when $\alpha > \beta > \delta$, $\alpha \geqq \gamma > \delta$, and $\mu \leqq \delta$.

It is clear that: $(\gamma \geqq \gamma' > \delta$ and $\mu' \leqq \mu$ and $\alpha' \geqq \alpha$ and $(\alpha,\beta) \xrightarrow[<\mu]{} (\gamma,\delta))$ implies that $(\alpha',\beta) \xrightarrow[<\mu']{} (\gamma',\delta)$. Note also that $(\alpha,\beta) \twoheadrightarrow (\gamma,\omega_0)$ and $\beta' \leqq \beta$ implies that $(\alpha,\beta') \twoheadrightarrow (\gamma,\omega_0)$.

Hence, when not asking for gaps on the left, the weakest form of the conjecture is of the following type: $(\alpha,\beta) \twoheadrightarrow (\delta^+,\delta)$.

One can, as in [9], equally consider other definitions, like those of $(\alpha,\beta) \xrightarrow[<\mu]{} (\gamma, \leqq \delta)$ and of $(\alpha,\beta) \xrightarrow[<\mu]{} (\gamma, <\delta)$. One can also consider transfer properties with several distinguished predicates, also an infinite number of them.

Now, in order to posit the problems, we shall recall a certain number of known results.

2. Known results

We recall the following:

PROPOSITION 2 (ZFC). *Assume that κ, α are infinite cardinals, such that $\alpha < \kappa$, and that κ is the Erdös cardinal $\mathscr{E}(\alpha)$. Take infinite cardinals λ, β, δ, μ, such that $\lambda \geqq \kappa > \beta > \delta$, $\alpha > \delta$ and $\mu \leqq \delta$. Then*

$$(\lambda,\beta) \xrightarrow[\leq\mu]{} (\alpha,\delta).$$

PROOF. Let L be a Skolem language, such that $|L| = \mu$. Let $M = (\lambda, \beta, L^M)$ be an L-structure of type (λ,β). Set $L^+ = L \cup \{c_\xi / \xi < \beta\}$, and $M^+ = (\lambda, \beta, L^M, (\xi)_{\xi<\beta})$. Let $I \subset \kappa$ be a set of indiscernibles for M^+, such that $|I| = \alpha$, and I is *remarkable*. It is clear that $I \subset \kappa - \beta$. Let $D \subset \beta$ be such that $|D| = \delta$. Set $X = \mathrm{Hull}(I \cup D, M)$, and $N = M \restriction_X$. It is plain that $|X| = \alpha$, and that $|X \cap \beta| \geqq \delta$, since $D \subset X \cap \beta$. We have then to show that $|X \cap \beta| \leqq \delta$. Set $L_D = L \cup \{c_\xi / \xi \in D\}$. Then $|L_D| = \delta$. Let $t(v_1, \ldots, v_n)$ be a term in L_D, and let

$c_1 < \cdots < c_n$, $e_1 < \cdots < e_n$ be elements of I. Since I is remarkable and $I \cap \beta = \emptyset$, we get

$$t(c_1, \dots, c_n) < \beta \rightarrow t(c_1, \dots, c_n) = t(e_1, \dots, e_n).$$

Hence $|X \cap \beta| \leqq |L_D| \leqq \delta$. QED Proposition 2

We now recall the following:

THEOREM 3 (Silver). *Assume that: "ZFC + $\mathscr{E}(\omega_1)$ exists" is consistent. Then, so is: ZFC + $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega_0)$.*

Donder proved a converse to Theorem 3 (see [3], theorem 1.7), namely:

THEOREM 4 (Donder). *Assume that $V \models ZFC + (\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega_0)$. Set $\lambda = \omega_2$, $\kappa = \omega_1$. Then, in K, we have $\lambda \rightarrow (\kappa)^{<\omega}$.*

It is immediate that the proof of Theorem 3 gives the following:

THEOREM 5. *Assume that $V \models \mathscr{E}(\omega_1)$ exists and MA and $2^\omega = \omega_2$ and $(\forall \tau \geqq \omega_2)(2^\tau = \tau^+)$. Take β regular, such that $\kappa > \beta > \omega_1$, where we set $\kappa = \mathscr{E}(\omega_1)$. Let P be the Levy–Solovay collapse, making $\kappa = \beta^+$. Then, in V^P, we have $(\beta^+, \beta) \twoheadrightarrow (\omega_1, \omega_0)$.*

As an immediate consequence, we get:

COROLLARY 6. Cons$(ZFC + \mathscr{E}(\omega_1)$ *exists*) *implies*

$$\text{Cons}(ZFC + (\aleph_{\omega+2}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega_0)).$$

PROOF. In Theorem 5, set $\beta = \aleph_{\omega+1}$. Hence, we get a model M such that $M \models (\aleph_{\omega+2}, \aleph_{\omega+1}) \twoheadrightarrow (\omega_1, \omega_0)$. Due to a previous remark, $M \models (\aleph_{\omega+2}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega_0)$. □

REMARK. $M \models 2^\omega = \omega_2$. Hence it remains to know whether $2^\omega = \omega_1 \wedge (\aleph_{\omega+2}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega_0)$ is consistent, and whether it is a "large cardinal" property. The same question is of course relevant for $(\beta^+, \beta) \twoheadrightarrow (\omega_1, \omega_0) \wedge \beta \geqq \omega_2 \wedge 2^\omega = \omega_1$, β regular.

It can be shown that, in fact, one can get the same transfer property, together with the GCH, by simply collapsing ω_2 over ω_1.

At last, let us quote the following (see [4]):

THEOREM 7 (Donder–Koepke). *Assume that $\tau \geqq \omega_1$, and that $(\tau^{++}, \tau^+) \twoheadrightarrow (\tau^+, \tau)$. Then 0^+ exists.*

(In fact, Donder and Koepke deduced the existence of 0^+ from a weaker hypothesis, the so-called "weak Chang Conjecture" for τ^+.)

3. Comments

(1) Due to Proposition 2 and Theorem 5, we see that $(\kappa,\lambda)\twoheadrightarrow(\alpha,\omega_0)$ cannot, in itself, imply the existence of a rather large cardinal, even if $\alpha \geqq \omega_2$. In fact, as we shall see later, in this second case, it comes from the fact that the gap on the left side is too large.

(2) Theorem 5 shows that the conjecture $(\beta^+,\beta)\twoheadrightarrow(\omega_1,\omega_0)$, for β regular (either successor or inaccessible) is also unable to imply the existence of larger cardinals, even if $\beta \geqq \omega_2$.

Hence we are led to the following questions:

QUESTION 1. Does $(\kappa^{++},\kappa)\twoheadrightarrow(\omega_2,\omega_0)$ imply the existence of a large cardinal?

QUESTION 2. Does $(\beta^+,\beta)\twoheadrightarrow(\omega_1,\omega_0)\wedge\beta$ is singular imply the existence of a large cardinal?

QUESTION 3. Does $(\kappa^+,\kappa)\twoheadrightarrow(\omega_2,\omega_1)$ imply the existence of a large cardinal?

(There are also variants of these questions.)

We shall now, in Part B, prove a general theorem, which shows that, in the three cases, 0^+ exists.

We are also interested in the consistency of the statements involved. Remarks which are almost immediate applications of known results will be made in Part B. Consistency results involving singular cardinals will be found in another paper.

Part B

4. $\mathrm{CC}(\lambda,\kappa)$

We shall devise a convenient form of the Conjecture of Chang, let us say $\mathrm{CC}(\lambda,\kappa)$, such that $(\exists\kappa)\mathrm{CC}(\kappa^+,\kappa)\Rightarrow\exists 0^+$, and such that any of the hypotheses invoked in the questions previously raised implies that $(\exists\kappa)\mathrm{CC}(\kappa^+,\kappa)$.

Let us first set:

DEFINITION 8. $\overline{\mathrm{CC}}(\lambda,\kappa)$ if and only if:

(a) λ and κ are two infinite cardinals, such that $\lambda\geqq\kappa$.

(b) For every finitary structure $M=(\lambda,\kappa,A)$ there exist $N=(X,X\cap\kappa,A\restriction_X)$, such that:

(1) $N\lhd M$,

(2) $|X|>|X\cap\kappa|$,

(3) $\kappa\not\subset X$.

Note that $\overline{\mathrm{CC}}(\lambda, \kappa)$ implies that $\lambda > \kappa \geqq \omega_1$, but it does not preclude our structure N from being such that $|X \cap \kappa| = \omega$.

($\kappa \geqq \omega_1$, for, by definability of every integer, $\omega \subset X$; $\lambda > \kappa$, for $|X| > |X \cap \kappa|$.)

Our aim is then to prove:

THEOREM 9. *Assume that, for some* singular *cardinal* κ, $\overline{\mathrm{CC}}(\kappa^+, \kappa)$ *is true. Then* 0^+ *exists.*

If κ is regular, $\overline{\mathrm{CC}}(\kappa^+, \kappa)$ does not even imply the existence of a Ramsey cardinal, as seen in Part A, for $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega_0)$ implies $\overline{\mathrm{CC}}(\kappa^+, \kappa)$. Hence, we have to add a condition to $\overline{\mathrm{CC}}(\lambda, \kappa)$, in order to be able to treat the cases in which κ is regular. Hence, we give the following definition:

DEFINITION 10. $\mathrm{CC}(\lambda, \kappa)$ if and only if λ, κ are infinite cardinals such that, for every finitary structure $M = (\lambda, \kappa, A)$, there exists $N = (X, X \cap \kappa, A \restriction_X)$, such that:

(1) $N \lhd M$,

(2) $|X| > |X \cap \kappa|$,

(3) $\kappa \not\subset X$,

(4) *If* κ is regular, *then* $|X| \geqq \omega_2$.

It is clear that condition (4) excludes the case $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega_0)$ and κ regular.

THEOREM 11. *Assume that, for some* κ, $\mathrm{CC}(\kappa^+, \kappa)$. *Then* 0^+ *exists.*

It is clear that Theorem 11 implies Theorem 9. We gave the definition piecewise in order to be more understandable.

Note also that $\mathrm{CC}(\kappa^+, \kappa)$ implies that $\kappa \geqq \omega_2$. For, $\mathrm{CC}(\omega_2, \omega_1)$ implies, due to condition (4) of the definition, that ω_2 is a Jonsson cardinal, which is known to be false.

Before proceeding to the proof of Theorem 11, we shall recall some facts about K, $L(\mu)$ and 0^+. We shall, as much as possible, rely on [4].

5. K, $L(\mu)$ and 0^+

(1) K denotes the core model.

(2) Assume that some ordinal is a measurable cardinal in some inner model. Let ρ denote the *least* such ordinal and let μ denote the unique object ν such that, in $L(\nu)$, ν is a normal ultrafilter over ρ. (ρ, μ) is then well-defined.

(3) "There exists 0^+" means that:

(a) $L(\mu)$ exists.

(b) $L(\mu)$ admits a closed and cofinal class of indiscernibles which generates it.

Hence, 0^+ is $L(\mu)^\#$, the theory of $L(\mu)$ with the unique *remarkable* class of indiscernibles which generates it.

(4) Note that the following is true:

LEMMA 12. *Let X be a definable class, which is not a set, such that $X \lhd L(\mu)$. Let A be a transitive class and let $\pi : A \xrightarrow[\sim]{} X$ be an isomorphism. Then $A = L(\mu)$.*

PROOF. We have $A = L(\bar{\mu})$ for some $\bar{\mu}$ over some $\bar{\rho}$. Clearly, $\bar{\rho} \leqq \rho$. By minimality of ρ, $\bar{\rho} = \rho$. By unicity of μ, $\bar{\mu} = \mu$. □

(5) Hence, we get

PROPOSITION 13. *Assume that there exists a non-trivial elementary embedding $j : L(\mu) \to L(\mu)$. Then 0^+ exists.*

PROOF. The proof goes like the one of Kunen's theorem for L and $0^\#$, for the only thing which is needed is the condensation lemma for $L(\mu)$, whose validity is asserted by Lemma 12. □

(6) We recall also that $L(\mu)$ exists if and only if there exists a non-trivial elementary embedding $j : K \to K$. Moreover, we have the following additional information (see [4], section 1.5):

PROPOSITION 14. *Let $j : K \to K$ be a non-trivial elementary embedding, and let γ be the critical point of j. Then:*

(a) *if $\omega_1 \leqq \gamma$, then $\rho < \gamma^+$,*

(b) *if $\gamma < \omega_1$, then $\rho \leqq \omega_1$.*

6. How to obtain $L(\mu)$

For any undefined concept or any missing proof in this section, we refer the reader to [4].

ZFC^- denotes ZFC minus the power set axiom.

Let us begin with a definition.

DEFINITION 15. Let A be a class. A is said to be *absolute for premice* iff: $M \in A$ and $A \models$ "M is an *iterable* premouse" implies that $V \models$ "M is an iterable premouse".

Now, establish a condensation lemma for K, which is an easy generalisation of Lemma 2.6 of [4].

Let M be an iterable premouse. Let γ denote the measurable cardinal in the sense of M (we say that M is a premouse *at* γ). $\mathrm{lp}(M)$ (the low part of M) denotes the set $H_\gamma^M = \{a \in M / M \models |TC(a)| < \gamma\}$. It is known that $\mathrm{lp}(M) = V_\gamma \cap M$.

For $i \in \mathrm{Ord}$, M_i denotes the ith iterate of M. Hence:

LEMMA 16. *Assume that A is* transitive, *that $A \models ZFC^- + V = K$, and that A is absolute for premice. Take an iterable premouse M at an ordinal γ, such that $\gamma \in A$. Then*:

$$\mathcal{P}(\gamma) \cap A \not\subset M \rightarrow \mathrm{lp}(M) \subset A.$$

REMARK. The point of this lemma is the following. Assume that A is obtained as a transitive collapse, of an elementary substructure, either of K or of K_λ, for λ some cardinal in the sense of K (recall that, whenever λ is a cardinal *in* K, we denote by "K_λ" the set H_λ^K; it is known that $K_\lambda \models \mathrm{ZFC}^- + V = K$). We want to know how successful we have been in condensation, i.e., how much of K is retained in A. We remember that K is the union of all $\mathrm{lp}(M)$, where M is an iterable premouse. Hence, Lemma 16 gives an answer to this question.

PROOF OF LEMMA 16. Take $a \in A$, $a \subset \gamma$, such that $a \notin M$. Two cases are possible:

(a) $a \in L$. Hence, there exists $i \in \mathrm{Ord}$, such that $a \in L_i$. But $L_i \subset M_i$. So $a \in M_i$. Hence $a \in \mathcal{P}(\gamma) \cap M_i = \mathcal{P}(\gamma) \cap M$. Hence $a \in M$, a contradiction. Hence, we have:

(b) $a \notin L$. Due to $A \models \mathrm{ZFC}^- + V = K$, find a premouse N in the sense of A, such that $a \in \mathrm{lp}(N)$. By absoluteness, $V \models$ "N is an iterable premouse". Iterating N in A if necessary, we can assume, since $\gamma \in A$, that N is a premouse at some ordinal ξ such that $\gamma \leqq \xi$. Take, in V, a regular cardinal θ, such that $\theta > |N|, |M|$. We know that three cases are possible:

$$\left.\begin{array}{l} N_\theta \in M_\theta \\ N_\theta = M_\theta \end{array}\right\} \Rightarrow N_\theta \subset M_\theta,$$

$M_\theta \in N_\theta$.

(c) $N_\theta \subset M_\theta$ *is impossible.* For, assume that $N_\theta \subset M_\theta$. Hence

$$a \in \mathcal{P}(\gamma) \cap N \subset \mathcal{P}(\xi) \cap N = \mathcal{P}(\xi) \cap N_\theta \subset \mathcal{P}(\xi) \cap M_\theta.$$

Hence $a \in M_\theta$. Hence $a \in M$.

(d) So $M_\theta \in N_\theta$. Hence $M_\theta \subset N_\theta$. But $\gamma \leqq \xi$. Hence

$$\mathrm{lp}(M) = V_\gamma \cap M = V_\gamma \cap M_\theta \subset V_\gamma \cap N_\theta \subset V_\xi \cap N_\theta = V_\xi \cap N \subset N \subset A.$$

QED Lemma 16

The application of Lemma 16 which is adequate for our purposes is the following:

LEMMA 17. *Assume that A is transitive, $A \models \mathrm{ZFC} + V = K$, and A is absolute for premice. Assume that $\lambda = A \cap \mathrm{Ord}$ is a cardinal in K. Assume, moreover, that $A \models$ "there exists a largest cardinal". Then $A = K_\lambda$.*

PROOF. (1) $A \subset K_\lambda$. Take $x \in A$ and $a \in A$ such that a is transitive and $x \in a$ (possible, since $a \models ZF^-$). Let α denote the largest cardinal of A. Take $f \in A$, such that f is a surjection, $f : \alpha \to a$ (still possible, because of the axiom of choice in A). Take $M \in A$ an iterable premouse in the sense of A, such that $f \in \mathrm{lp}(M)$. M is an iterable premouse in V, so $\mathrm{lp}(M) \subset K$. Hence $f \in K$. So $a \in K$, and $K \models |a| \leqq |\alpha| < \lambda$.

(2) $K_\lambda \subset A$. Take $x \in K_\lambda$, and take M, an iterable premouse in V, such that $x \in \mathrm{lp}(M)$ and $M \in K_\lambda$ (this is possible, even if $\lambda < \omega_1^V$). Let α denote the largest cardinal of A, and let γ be the ordinal such that M is a premouse at γ. We may as well assume that $\alpha < \gamma < \lambda$, iterating M in K_λ if necessary. Since $\lambda \subset A$, we get $\gamma \in A$. But $\mathcal{P}(\gamma) \cap A \not\subset M$, since there is a bijection $f : \alpha \longrightarrow \gamma$, $f \in A$. Hence, by Lemma 16, $x \in \mathrm{lp}(M) \subset A$. QED Lemma 17

We need one more lemma.

LEMMA 18. *Let A and B be two transitive models of ZFC^-. Let $\pi : A \to B$ be a non-trivial elementary embedding. Let γ be the critical point of π, and assume that $K_{\gamma^+} \subset A$. Then, there exists a non-trivial elementary embedding $j : K \to K$, such that $\mathrm{cp}(j) = \gamma$.*

(Hence, by (6) in Section 5, $L(\mu)$ exists. Moreover, by Proposition 14:
(a) $\omega_1 \leqq \gamma \to \rho < \gamma^+$,
(b) $\gamma < \omega_1 \to \rho \leqq \omega_1$.)

PROOF OF LEMMA 18. The proof goes as usual. Due to the hypothesis, $K_{(\gamma^+ K)} \subset A$. Hence $\mathcal{P}(\gamma) \cap K \subset A$. So define an ultrafilter D over $\mathcal{P}(\gamma) \cap K$, as follows: Take $X \subset \gamma$ such that $X \in K$. Then $X \in D \leftrightarrow \gamma \in \pi(X)$. We have only to show that $(K^\gamma \cap K)/D$ is well-founded. Assume not. Take $(f_n)_{n<\omega}$ a bad sequence and, for $n < \omega$, set

$$X_n = \{\alpha < \gamma / f_{n+1}(\alpha) \in f_n(\alpha)\}.$$

Hence $n < \omega \to X_n \in D$. Take τ regular in V such that $\tau \geqq \omega_2$ and $Z = \{f_n / n < \omega\} \cup (\gamma + 1) \subset K_\tau$. Set $H = \mathrm{Hull}(Z, K_\tau)$. Hence $|H| = |\gamma|$. Take T transitive and $h : H \longrightarrow T$ an isomorphism. T is absolute for premice, and $|T| < \gamma^+$. Hence

$T \subset K_{\gamma^+} \subset A$. (For, take $x \in T$. Hence, there exists a premouse M in the sense of T such that $x \in \mathrm{lp}(M)$. Since M is in T, we have $|M| < \gamma^+$. Hence $\mathrm{lp}(M) \subset K_{\gamma^+}$.) But $n < \omega \to h(f_n) = h \circ f_n$. So

$$n < \omega \to X_n = \{\alpha < \gamma / h(f_{n+1})(\alpha) \in h(f_n)(\alpha)\}.$$

Take $n < \omega$. Since $X_n \in D$, we get $j(h(f_{n+1}))(\gamma) \in j(h(f_n))(\gamma)$, contradicting the well-foundeness of B. QED Lemma 18

7. How to obtain 0^+

Assume that $L(\mu)$ exists. Let (ρ, μ) be as usual. We need the following:

LEMMA 19. *Assume that A and B are transitive, satisfy ZFC^-, that $L(\mu)$ exists, and that there exists an elementary embedding $\pi : A \to B$, such that:*

(1) *π is non-trivial.*

(2) $\pi(\rho) = \rho$.

(3) *Let γ be the critical point of π. Set: $\theta = \sup(\rho, \gamma)$. Then $L_{\theta^+}(\mu) \subset A$.*

Then 0^+ *exists.*

PROOF. Due to Proposition 13, we have only got to find a non-trivial elementary embedding $j : L(\mu) \to L(\mu)$.

We recall a theorem of Silver, which says that, for $\alpha \geqq \rho$, $\mathscr{P}(\alpha) \cap L(\mu) \subset L_{\alpha^+}(\mu)$. So let γ be the critical point of π. Set $\theta = \sup(\rho, \gamma)$. We get

$$\mathscr{P}(\gamma) \cap L(\mu) \subset \mathscr{P}(\theta) \cap L(\mu) \subset L_{\theta^+}(\mu)$$

(since $\theta \geqq \rho$) $\subset A$. Hence we can define an ultrafilter D over $\mathscr{P}(\gamma) \cap L(\mu)$ as follows: for $X \subset \gamma$, $X \in L(\mu)$, set $X \in D \leftrightarrow \gamma \in \pi(X)$. Set $N = (L(\mu)^\gamma \cap L(\mu))/D$, and let $j : L(\mu) \to N$ be the associated elementary embedding.

(1) *N is well-founded.* For, if not, we would proceed as usual and, using the argument of Silver again, contradict the well-foundness of B.

(2) Hence $N = L(j\mu)$, where $j\mu$ is, in $L(j\mu)$, a normal ultrafilter over $j\rho$. We have got to show that $j\rho = \rho$, showing hence that $N = L(\mu)$. So

(3) $j\rho = \rho$. Let us observe that $L_{j(\theta^+)}(j\mu)$ is isomorphic to $(L_{\theta^+}(\mu)^\gamma \cap L(\mu))/D$. By the argument of Silver again, $L_{\theta^+}(\mu)^\gamma \cap L(\mu) = L_{\theta^+}(\mu)^\gamma \cap L_{\theta^+}(\mu)$. So we can define as follows a map $k : L_{j(\theta^+)}(j\mu) \to B$. Take $f \in L_{\theta^+}(\mu)^\gamma \cap L_{\theta^+}(\mu)$; set $k([f]_D) = \pi(f)(\gamma)$. Clearly, k is elementary. Moreover,

$$\pi \restriction L_{\theta^+}(\mu) = k \circ j \restriction L_{\theta^+}(\mu).$$

Hence $\rho \leqq j(\rho) \leqq k(j(\rho)) = \pi(\rho) = \rho$, as was to be shown. QED Lemma 19

8. Proof of Theorem 11

So let us now prove Theorem 11.

Assume that κ is an infinite cardinal, and $CC(\kappa^+, \kappa)$ holds. Hence, as noticed, $\kappa \geqq \omega_1$.

8.1. *The existence of $L(\mu)$*

We shall show the following:

CLAIM 1. *$L(\mu)$ exists, and $\rho < \kappa^+$.*

To prove Claim 1, we shall distinguish three cases.

Case 1: κ is singular and $\kappa^{+K} < \kappa^+$.

Hence, by the covering lemma for K (see [4], Lemma 2.4(1) and Lemma 2.3), $L(\mu)$ exists and $\rho < \kappa^+$.

Case 2: $\kappa^{+K} = \kappa^+$.

Set $M = (K_{\kappa^+}, \in, \kappa, \boldsymbol{\kappa})$, where $\boldsymbol{\kappa}$ is a constant, intended to denote κ, while "κ" is the realisation in M of an unary predicate, say a. Due to $CC(\kappa^+, \kappa)$, take $N = (X, \in, X \cap \kappa, \boldsymbol{\kappa})$, such that:

(1) $N \lhd M$,

(2) $|X \cap \kappa| < |X|$,

(3) $\kappa \not\subset X$.

Take a transitive set A, and an isomorphism $\pi : \boldsymbol{A} \longrightarrow N$, where $\boldsymbol{A} = \pi^{-1}(N)$. Set $\alpha = \pi^{-1}(\kappa)$. It is clear that $\alpha = \mathrm{ot}(X \cap \kappa) = \pi^{-1}(X \cap \kappa)$. Hence, by (2), we get that $|A| = |X| > |X \cap \kappa| \geqq \alpha$. So $|A| \geqq \alpha^+$. Hence $\mathrm{Ord} \cap A \geqq \alpha^+$.

SUBCLAIM 2. *π is non-trivial. Let γ be the critical point of π. Then $\gamma \leqq \alpha$.*

PROOF OF SUBCLAIM 2. Assume that $\alpha < \gamma$, setting $\gamma = \mathrm{Ord} \cap A$ if π is trivial. Set $\boldsymbol{A} = (A, \in, \alpha, \boldsymbol{\alpha})$, where $\alpha = \pi^{-1}(X \cap \kappa)$. By definition,

$$\xi \in \alpha \leftrightarrow \boldsymbol{A} \models a(\xi) \leftrightarrow N \models a(\pi(\xi)) \leftrightarrow \pi(\xi) \in X \cap \kappa.$$

But $\alpha \leqq \gamma$ gives $\xi < \alpha \rightarrow \pi(\xi) = \xi$. Hence $\xi \in \alpha \leftrightarrow \xi \in X \cap \kappa$. Hence $X \cap \kappa = \alpha$, and $\alpha \subset X$. But since $\alpha < \gamma$, we get $\alpha = \pi(\alpha) = \kappa$. So $\alpha = \kappa$. Hence $\kappa \subset X$, contradicting (3). QED Subclaim 2

SUBCLAIM 3. $A = K_{\alpha^+}$.

PROOF OF SUBCLAIM 3. Since κ is the largest cardinal in M (due to "$\kappa^{+K} = \kappa^+$"), we have that: $\boldsymbol{A} \models$ "α is the largest cardinal". But hence $\mathrm{Ord} \cap A \leqq \alpha^+$, since, if $\alpha^+ < \mathrm{Ord} \cap A$, α^+ would be a cardinal in $\boldsymbol{A}$, with $\alpha < \alpha^+$. So we get $\mathrm{Ord} \cap A = \alpha^+$. Hence we can apply Lemma 17, which tells us that $A = K_{\alpha^+}$.

QED Subclaim 3

Let us now finish the proof of Case 2.

We apply Lemma 18 to π, setting $B = K_{\kappa^+}$. Since $A = K_{\alpha^+}$, we get $K_{\gamma^+} \subset A$. Hence we conclude that:

(1) $L(\mu)$ exists,

(2) (a) $\omega_1 \leqq \gamma \rightarrow \rho < \gamma^+$,

(b) $\gamma < \omega_1 \rightarrow \rho \leqq \omega_1$.

Let us show that, in both cases, $\rho < \kappa^+$.

Case (a): $\rho < \gamma^+ \leqq \alpha^+ \leqq \kappa^+$.

Case (b): $\rho \leqq \omega_1 \leqq \kappa < \kappa^+$. QED Case 2

Case 3: κ is regular and $\kappa^{+K} < \kappa^+$.

We shall construct our structure, using an idea of [4]. Set

$$S = \{\xi / \kappa < \xi < \kappa^+ \text{ and } \operatorname{cof}(\xi) = \kappa\}.$$

For $\xi \in S$, let $G_\xi : \kappa \rightarrow \xi$ be a *normal cofinal* function. Set $G = \bigcup_{\xi \in S} \{\xi\} \times G_\xi$. Finally, set $M = (K_{\kappa^+}, \in, \kappa, \boldsymbol{\kappa}, G)$. Due to $\mathrm{CC}(\kappa^+, \kappa)$, let $N = (X, \in, X \cap \kappa, \boldsymbol{\kappa}, G \restriction_X)$ be such that

(1) $N \lhd M$,

(2) $|X \cap \kappa| < |X|$,

(3) $\kappa \not\subset X$,

(4) $|X| \geqq \omega_2$.

Let $\boldsymbol{A}$ be the transitive collapse of N. Set $\boldsymbol{A} = (A, \in, \alpha, \boldsymbol{\alpha}, F)$, where $\alpha = \operatorname{ot}(X \cap \kappa)$. Let $\pi : \boldsymbol{A} \xrightarrow{\sim} N$ be the isomorphism. Hence $\pi(\alpha) = \kappa$, and $\alpha = \pi^{-1}(X \cap \kappa)$. Clearly $\alpha^+ \leqq \mathrm{Ord} \cap A$.

Subclaim 4. *π is non-trivial.* $\mathrm{Cp}(\pi) \leqq \alpha$.

The proof is given by the one of Subclaim 2, for this one did not use the fact that $\kappa^{+K} = \kappa^+$.

Subclaim 5. $K_{\alpha^+} \subset A$.

Proof of Subclaim 5. Take $x \in K_{\alpha^+}$. We have to show that $x \in A$. Take M, a premouse in the sense of K_{α^+}, such that $x \in \mathrm{lp}(M)$. Assume that M is a premouse at some ordinal ξ. We can clearly assume that $\alpha < \xi < \alpha^+$.

Let $(M_i, \xi_i)_{i \in \mathrm{Ord}}$ denote the iteration of (M, ξ). Set $\lambda = \mathrm{Ord} \cap A$. Since $|X| \geqq \omega_2$, we have $\lambda \geqq \omega_2 > \omega_1$. Hence:

(a) Either $\alpha^+ > \omega_1$, and hence $\xi_{\omega_1} < \alpha^+ \leqq \lambda$,

(b) or $\alpha^+ = \omega_1$, and hence $\xi_{\omega_1} = \alpha^+ < \omega_2 \leqq \lambda$.

So, in both cases, $\xi_{\omega_1} < \lambda$. (Let us note that this is the only point of the proof at which we use the hypothesis $|X| \geqq \omega_2$.)

Hence $\xi_{\omega_1} \in A$. So define $\bar{\xi}$ as follows:

$$\bar{\xi} = \begin{cases} \xi_\omega & \text{if } V \models \operatorname{cof}(\alpha) \neq \omega, \\ \xi_{\omega_1} & \text{if } V \models \operatorname{cof}(\alpha) = \omega. \end{cases}$$

Hence $V \models \operatorname{cof}(\bar{\xi}) \neq \operatorname{cof}(\alpha)$. Moreover, $\bar{\xi} \in A$. But then $\boldsymbol{A} \models$ "$F_{\bar{\xi}}$ is undefined", for, if $F_{\bar{\xi}}$ would be defined, it would be a normal cofinal map $\alpha \to \bar{\xi}$, implying that $V \models \operatorname{cof}(\alpha) = \operatorname{cof}(\bar{\xi})$. Hence $M \models$ "$G_{\pi(\bar{\xi})}$ is undefined". Hence, $V \models \operatorname{cof}(\pi(\bar{\xi})) < \kappa$, due to the definition of G. Applying Lemma 2.4(2) and Lemma 2.3 of [4] (the former applying to our case, since $\kappa \geqq \omega_2$, by a previous remark), we see that:

(a) either ($L(\mu)$ exists and $\rho < \kappa^+$),

(b) or $\pi(\bar{\xi})$ is singular *in* K_{κ^+}.

So assume that we are in case (b). By the elementarity of π, $\boldsymbol{A} \models$ "$\bar{\xi}$ is singular". Take $\tau < \bar{\xi}$, and take $f \in A$, such that $f : \tau \to \bar{\xi}$ is cofinal. Let $\bar{M}$ be the iterate of M corresponding to $\bar{\xi}$. Clearly, $f \in \mathcal{P}(\bar{\xi}) \cap A$, and $f \notin M$. Hence, due to Lemma 16, we see that $\operatorname{lp}(\bar{M}) \subset A$. But hence $\operatorname{lp}(M) \subset \operatorname{lp}(\bar{M}) \subset A$, as was to be shown. QED Case 3 — QED Claim 1

8.2. The existence of $0^\dagger$

The various cases are treated at once.

We have shown that $L(\mu)$ exists, and that $\rho < \kappa^+$. So take a surjection $f : \kappa \to \rho$, $f \in V$. Set $M = (L_{\kappa^+}(\mu), f, \in, \kappa, \boldsymbol{\kappa}, \mu,)$. Take $N = (X, f\restriction_X, \in, X \cap \kappa, \underline{\kappa}, \mu \cap X)$ such that:

(1) $N \lhd M$,

(2) $|X| > |X \cap \kappa|$,

(3) $\kappa \not\subset X$.

(Note that we no longer need the condition: if κ is regular, then $|X| \geqq \omega_2$.) Take $\pi : \boldsymbol{A} \xrightarrow{\sim} N$, a transitive collapse. Set

$$\alpha = \pi^{-1}(X \cap \kappa) = \pi^{-1}(\boldsymbol{\kappa}) = \operatorname{ot}(X \cap \kappa).$$

Set $\boldsymbol{A} = (L_\lambda(\nu), g, \in, \alpha, \boldsymbol{\alpha}, \nu)$, for some ordinals λ, $\bar{\rho}$, such that $\nu \subset \mathcal{P}(\bar{\rho})$ and $\pi(\bar{\rho}) = \rho$. By (2), $\alpha^+ \leqq \lambda$. But $\boldsymbol{A} \models$ "$g : \alpha \to \bar{\rho}$ is onto". Hence $\bar{\rho} < \alpha^+$. Hence, by the argument of Silver again, $\mathcal{P}(\bar{\rho}) \cap L(\nu) \subset L_{\bar{\rho}^+}(\nu) \subset L_{\alpha^+}(\nu) \subset L_\lambda(\nu) = A$, since $\nu \subset \mathcal{P}(\bar{\rho})$. Hence ν is a normal measure at $\bar{\rho}$ *in* $L(\nu)$. But $\bar{\rho} \leqq \rho$. By minimality of ρ, we get $\bar{\rho} = \rho$, and, hence, $\pi(\rho) = \rho$. By unicity of ν, we have $\mu = \nu$. Since $\rho = \bar{\rho}$, we have $\rho < \alpha^+$. On the other hand, $\gamma \leqq \alpha < \alpha^+$. Hence, setting $\theta =$

$\sup(\rho, \gamma)$, we get $\theta < \alpha^+$, so $L_{\theta^+}(\mu) \subset L_{\alpha^+}(\mu) \subset A$. Moreover $\pi(\rho) = \rho$. Hence the conditions of Lemma 19 are fulfilled, and 0^+ exists.

QED 8.2

QED Theorem 11

9. Applications

THEOREM 20. *Let κ and α be two infinite cardinals such that $\kappa > \alpha$. Each of the following hypotheses implies the existence of 0^+:*

(1) $(\kappa^+, \kappa) \twoheadrightarrow (\alpha^+, \alpha)$ *and* $\alpha \geqq \omega_1$,

(2) $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega_0)$ *and κ is singular.*

PROOF. It is enough to show that each of the hypotheses implies $\mathrm{CC}(\kappa^+, \kappa)$.

So let $M = (\kappa^+, \kappa, A)$ be a structure of type (κ^+, κ) and $N = (X, X \cap \kappa, A \restriction_X) \lhd M$ be given either by (1) or (2). In both cases, $|X \cap \kappa| < |X|$. Moreover $\kappa \not\subset X$, since, if $\kappa \subset X$, then $\kappa \subset X \cap \kappa$, so $|\kappa| \leqq \alpha$, so $\kappa = \alpha$, a contradiction.

Now assume that κ is regular. Then $|X| \geqq \alpha^+ \geqq \omega_2$. QED Theorem 20

10. More corollaries

We would like to show that some other instances of the Conjecture of Chang imply the existence of 0^+, for instance, that it follows from $(\kappa^{++}, \kappa) \twoheadrightarrow (\alpha^{++}, \alpha)$ for any pair (κ, α), such that $\kappa > \alpha \geqq \omega$, hence, for instance, from $(\omega_3, \omega_1) \twoheadrightarrow (\omega_2, \omega_0)$, as well as from $(\omega_4, \omega_2) \twoheadrightarrow (\omega_2, \omega_0)$, or from $(\omega_5, \omega_3) \twoheadrightarrow (\omega_2, \omega_0)$.

Hence, in order to evaluate the strength of a hypothesis of the form $(\lambda, \kappa) \twoheadrightarrow (\beta, \alpha)$, we recall a previous remark and make an additional one.

(1) $(\beta^+, \beta) \twoheadrightarrow (\omega_1, \omega_0)$ is quite weak, for β regular, since it has been seen in Part A not to be stronger than the existence of $\mathscr{E}(\omega_1)$.

(2) "$(\lambda^{++}, \lambda) \twoheadrightarrow (\omega_1, \omega_0)$ and λ is singular" is quite weak too, (Indeed, we have seen that, starting with $\mathscr{E}(\omega_1)$, one can get $(\aleph_{\omega+2}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega_0)$.)

Hence, for a larger-than-one gap hypothesis to imply the existence of 0^+, we can only look at the following: $(\lambda, \kappa) \twoheadrightarrow (\beta, \alpha)$, with

(a) either $\beta \geqq \alpha^{++}$,

(b) or $\beta = \alpha^+$ and $\alpha \geqq \omega_1$.

Clearly, $(\lambda, \kappa) \twoheadrightarrow (\beta, \alpha)$ and $\beta \geqq \alpha^{++}$ implies that $(\lambda, \kappa) \twoheadrightarrow (\alpha^{++}, \alpha)$. Hence we are left to look at:

(e) either $(\lambda, \kappa) \twoheadrightarrow (\alpha^{++}, \alpha)$, where $\kappa > \alpha \geqq \omega_0$,

(f) or $(\lambda, \kappa) \twoheadrightarrow (\alpha^+, \alpha)$, where $\kappa > \alpha \geqq \omega_1$.

Let us recall that, when the gap on the left is too large, the hypothesis becomes quite weak. Hence let us look at finite gaps on the left.

As to (e), let us quote the following:

LEMMA 21. *Let n be an integer such that $n \geqq 2$. Let κ and α be infinite cardinals such that $\kappa > \alpha$. Assume that $(\kappa^{+n}, \kappa) \twoheadrightarrow (\alpha^{+n}, \alpha)$. Take two integers i, j such that $0 \leqq i < j \leqq n$. Then $(\kappa^{+j}, \kappa^{+i}) \twoheadrightarrow (\alpha^{+j}, \alpha^{+i})$.*

PROOF. Assume that i, j are such integers, and that $(\kappa^{+j}, \kappa^{+i}) \not\twoheadrightarrow (\alpha^{+j}, \alpha^{+i})$. Let $M = (\kappa^{+j}, \kappa^{+i}, A)$ be a structure, without an elementary substructure of type $(\alpha^{+j}, \alpha^{+i})$. For $0 \leqq k < n$, let G_k be a family of functions showing that κ^{+k+1} is the successor of κ^{+k}. Set $M^+ = (\kappa, \kappa^{+1}, \ldots, \kappa^{+n}, G_0, G_1, \ldots, G_n, A)$. Let $N^+ = (X, \ldots)$ be an elementary substructure of M^+ such that $|X| = \alpha^{+n}$ and $|X \cap \kappa| = \alpha$. Due to the family (G_k), we see that, for $0 \leqq k < n$, $|X \cap \kappa^{+k+1}| \leqq |X \cap \kappa^{+k}|^+$. Hence we must have, for $0 \leqq k \leqq n$, $|X \cap \kappa^{+k}| = \alpha^{+k}$. So set $N = (X \cap \kappa^{+j}, X \cap \kappa^{+i}, X \cap A)$. It is plain that $N \lhd M$, a contradiction.

QED Lemma 21

As an immediate corollary, we get:

THEOREM 22. *Assume that n is an integer such that $n \geqq 2$, and κ, α are infinite cardinals such that $\kappa > \alpha$. Assume, moreover, that $(\kappa^{+n}, \kappa) \twoheadrightarrow (\alpha^{+n}, \alpha)$. Then 0^+ exists.*

PROOF. Due to Lemma 21, we have $(\kappa^{+n}, \kappa^{+n-1}) \twoheadrightarrow (\alpha^{+n}, \alpha^{+n-1})$, and we apply Theorem 20(1). □

As to (f), and in order to treat all finite gaps, let us quote the following.

LEMMA 23. *Assume that λ, κ, β, α are infinite cardinals such that $\kappa > \beta$ and $\alpha \geqq \beta$. Assume that $(\lambda^+, \kappa) \twoheadrightarrow (\alpha^+, \beta)$. Then, either $(\lambda^+, \lambda) \twoheadrightarrow (\alpha^+, \alpha)$ or $(\lambda, \kappa) \twoheadrightarrow (\alpha^+, \beta)$.*

PROOF. Assume, to the contrary, that both conclusions are false. Let $M_0 = (\lambda^+, \lambda, A)$ be a structure without elementary substructure of type (α^+, α) and $M_1 = (\lambda, \kappa, B)$ without elementary substructure of type (α^+, β). Let F be a family of functions showing that λ^+ is the successor of λ. Set $M = (\lambda^+, \lambda, \kappa, A, B, F)$, and let $N = (X, \ldots)$ be an elementary substructure of M, of type (α^+, β), i.e., such that $|X| = \alpha^+$ and $|X \cap \kappa| = \beta$. Due to F, we have $|X \cap \lambda| \geqq \alpha$. But $|X \cap \lambda| = \alpha$ is impossible, due to M_0, while $|X \cap \lambda| = \alpha^+$ is impossible, due to M_1. QED Lemma 23

As an immediate corollary, we get:

THEOREM 24. *Let κ and α be two infinite cardinals such that $\kappa > \alpha$ and n, m be two integers such that $2 \leqq m \leqq n$. Assume that $(\kappa^{+n}, \kappa) \twoheadrightarrow (\alpha^{+m}, \alpha)$. Then 0^+ exists.*

THEOREM 25. *Let κ, α be two infinite cardinals such that $\kappa > \alpha \geqq \omega_1$ and let n be an integer such that $n \geqq 1$. Assume that $(\kappa^{+n}, \kappa) \twoheadrightarrow (\alpha^{+}, \alpha)$. Then 0^+ exists.*

PROOF OF THEOREM 24. We apply Lemma 23 inductively. Since $(\kappa^{+n}, \kappa) \twoheadrightarrow (\alpha^{+m}, \alpha)$, we must have:

(a) either $(\kappa^{+n}, \kappa^{+n-1}) \twoheadrightarrow (\alpha^{+m}, \alpha^{+m-1})$, in which case 0^+ exists, due to Theorem 20(1),

(b) or $(\kappa^{+n-1}, \kappa) \twoheadrightarrow (\alpha^{+m}, \alpha)$.

If we stay in case (b), we must end with $(\kappa^{+m}, \kappa) \twoheadrightarrow (\alpha^{+m}, \alpha)$, which implies the existence of 0^+, due to Theorem 21. QED Theorem 24

PROOF OF THEOREM 25. We proceed in the same way. Either we get 0^+, or we get $(\kappa^{+n-1}, \kappa) \twoheadrightarrow (\alpha^{+}, \alpha)$. We end with $(\kappa^{+}, \kappa) \twoheadrightarrow (\alpha^{+}, \alpha)$, and still get 0^+. QED

This settles the problem for finite gaps, for the remaining cases have already been treated, one in Theorem 20(2), and the remaining ones (quite weak) in some remarks; concerning those, let us state, for the sake of completeness:

THEOREM 26. *The transfer hypothesis $(\kappa^{+n}, \kappa) \twoheadrightarrow (\omega_1, \omega_0)$ is not stronger (consistencywise) than the existence of $\mathscr{E}(\omega_1)$, in the following cases:*

(a) *κ is regular and $n \geqq 1$,*

(b) *κ is singular and $n \geqq 2$.*

We prove a last theorem, which is not without interest. To this end, let us modify the definition of $CC(\lambda, \kappa)$ and introduce a principle slightly different, denoted by $CC^+(\lambda, \kappa)$, which is obtained from $CC(\lambda, \kappa)$ by deleting condition (4), and replacing it by the following:

(4^+) $|X \cap \kappa| \geqq \omega_1$.

It is plain that $CC^+(\lambda, \kappa) \Rightarrow CC(\lambda, \kappa)$. Moreover:

LEMMA 27. *Assume that $\lambda > \mu > \kappa$. Then*

$$CC^+(\lambda, \kappa) \Rightarrow (CC^+(\lambda, \mu) \vee CC^+(\mu, \kappa)).$$

PROOF. Assume, to the contrary, the conclusion is false.

Let $M_0 = (\lambda, \mu, A)$ and $M_1 = (\mu, \kappa, B)$ be two structures witnessing this fact. Set $M = (\lambda, \mu, \kappa, A, B)$. Due to $CC^+(\lambda, \kappa)$, let $N = (X, \ldots) \lhd M$ be such that

(1) $\kappa \not\subset X$,

(2) $|X| > |X \cap \kappa|$,

(3) $|X \cap \kappa| \geqq \omega_1$.

It is clear that $(X, X \cap \mu, A \restriction_X) \lhd M_0$. Since $\kappa \not\subset X$, we get $\mu \not\subset X$. Since $|X \cap \kappa| \geqq \omega_1$, we get $|X \cap \mu| \geqq \omega_1$. Hence $|X| = |X \cap \mu|$. It is equally clear that $(X \cap \mu, X \cap \kappa, B \restriction_X) \lhd M_1$. Moreover $\kappa \not\subset X \cap \mu$ and $|X \cap \kappa| \geqq \omega_1$. Hence $|X \cap \mu| = |X \cap \kappa|$, hence $|X| = |X \cap \kappa|$, a contradiction. QED Lemma 27

As a corollary, we get:

THEOREM 28. *If* $1 \leqq n < \omega$ *and* $\mathrm{CC}^+(\kappa^{+n}, \kappa)$, *then* 0^+ *exists.*

REMARK. We can use the methods of the preceding lemmas to show that one still gets 0^+, with the assumption of some "small" infinite gaps on the left. For instance, from $(\lambda^{+\omega}, \lambda) \twoheadrightarrow (\alpha^{+i}, \alpha)$, one gets the existence of 0^+, whenever $1 \leqq i \leqq \omega$ and $\alpha \geqq \omega_1$, or $\alpha = \omega_0$ and $2 \leqq i \leqq \omega$. We can also, with methods analogous to the ones used in the proof of Theorem 11, deduce the existence of 0^+ from $\mathrm{CC}^+(\lambda, \kappa)$ for "reasonable" gaps between λ and κ.

REMARK. It is worth noting that, for example, for any integer $n \geqq 1$, $(\aleph_{\omega_1+n}, \aleph_{\omega_1}) \twoheadrightarrow (\omega_2, \omega_1)$ already implies the existence of 0^+, in contradiction to $(\aleph_{\omega_1+2}, \aleph_{\omega_1}) \twoheadrightarrow (\omega_1, \omega_0)$.

11. Some consistency results

Let us now concentrate our attention on consistency results of the following type:

(a) $(\lambda^+, \lambda) \twoheadrightarrow (\omega_2, \omega_1)$, where λ is regular and $\lambda > \omega_2$.

(b) $(\lambda^{++}, \lambda) \twoheadrightarrow (\omega_2, \omega_0)$, where λ is regular and $\lambda \geqq \omega_2$.

(It will be clear that we could replace $(\omega_0, \omega_1, \omega_2)$ by $(\alpha, \alpha^+, \alpha^{++})$, with the condition $\lambda > (\geqq) \alpha^+$.)

(Note that the consistency of, say, $(\omega_3, \omega_1) \twoheadrightarrow (\omega_2, \omega_0)$ has been established in [5].)

The consistency of (a) and (b) is obtained almost immediately, using known methods, due to Kunen. Let us, for example, show:

THEOREM 29. *Assume that there exists a non-trivial elementary embedding* $j: V \to M$ *such that* M *is transitive and that, setting* $\kappa = \mathrm{cp}(j)$, *we get* $M^{j(\kappa^+)} \subset M$. *Assume furthermore that* $V \models \mathrm{GCH}$. *Then, there exists a complete Boolean algebra in* V, *say* B, *such that, in* V^B, *we have* $(\omega_4, \omega_3) \twoheadrightarrow (\omega_2, \omega_1)$, *as well as* $(\omega_4, \omega_2) \twoheadrightarrow (\omega_2, \omega_0)$.

PROOF OF THEOREM 29. Set $\lambda = j(\kappa)$. We shall, using twice a Levy–Solovay collapse, make $\kappa = \omega_1$ and $\lambda = \omega_3$ hence having in V^B $\kappa = \omega_1$, $\kappa^+ = \omega_2$, $\lambda = \omega_3$ and $\lambda^+ = \omega_4$. We proceed indirectly, as in [7], constructing a "universal collapse". In the second step, we do not need a Silber collapse for we are not making λ the successor of κ in V^B. The properties invoked in the theorem then follow from the corresponding properties in V, namely $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ and $(\lambda^+, \kappa^+) \twoheadrightarrow (\kappa^+, <\kappa)$, as well as from a closure argument in the generic extension. QED

REMARK. Let n be an integer such that $n \geqq 1$. In the previous proof, we could have made $\kappa = \omega_n$, $\kappa^+ = \omega_{n+1}$, $\lambda = \omega_{n+2}$ and $\lambda^+ = \omega_{n+3}$, obtaining in V^B

$$(\omega_{n+3}, \omega_{n+2}) \xrightarrow[<\omega_n]{} \!\!\!\twoheadrightarrow (\omega_{n+1}, \omega_n)$$

as well as

$$(\omega_{n+3}, \omega_{n+1}) \xrightarrow[<\omega_n]{} \!\!\!\twoheadrightarrow (\omega_{n+1}, \omega_{n-1}).$$

REMARK. In the model given by Theorem 29 we also have

$$(\omega_4, \omega_1) \twoheadrightarrow (\omega_2, \omega_0), \qquad (\omega_3, \omega_1) \twoheadrightarrow (\omega_1, \omega_0),$$

and

$$(\omega_3, \omega_2) \twoheadrightarrow (\omega_1, \omega_0).$$

Add $n - 1$ to each index and obtain a statement which is true in the model of the preceding remark.

REMARK. An immediate variation gives, starting with GCH and a $j : V \to M$ such that $M^{j(\kappa^{++})} \subset M$, the consistency of $(\omega_6, \omega_5, \omega_4) \twoheadrightarrow (\omega_3, \omega_2, \omega_1)$, hence of $(\omega_6, \omega_4) \twoheadrightarrow (\omega_3, \omega_1)$, for example. Some transfer properties are valid in the same model, such as $(\omega_6, \omega_4, \omega_1) \twoheadrightarrow (\omega_3, \omega_1, \omega_0)$, hence $(\omega_6, \omega_1) \twoheadrightarrow (\omega_3, \omega_0)$. Varying the model, we can also add a fixed integer n to the indices of every statement.

Let us now concentrate on the model $M = V(G)$, given by Theorem 29, such that $M \models (\omega_4, \omega_3) \twoheadrightarrow (\omega_2, \omega_1)$, as well as the four other transfer properties.

THEOREM 30. $M \not\models (\omega_4, \omega_3) \twoheadrightarrow (\omega_3, \leqq \omega_2)$.

In order to prove Theorem 30, we shall recall some well-known facts.

DEFINITION 31. Let κ be a regular cardinal such that $\kappa \geqq \omega_1$. Let $A \subset \kappa$ be stationary in κ. Set $\diamondsuit^*(A)$ if and only if there exists a family $(S_\alpha)_{\alpha<\kappa}$ such that

(a) $\alpha < \kappa \rightarrow S_\alpha \subset \mathcal{P}(\alpha) \wedge |S_\alpha| \leqq \alpha$,

(b) For all $X \subset \kappa$, there exists a club $C \subset \kappa$ such that

$$C \cap A \subset \{\alpha < \kappa / X \cap \alpha \in S_\alpha\}.$$

We set $\diamondsuit^*_\kappa \leftrightarrow \diamondsuit^*(\kappa)$.

We recall:

PROPOSITION 32. *Assume that $\kappa = \rho^+$ where $\rho \geqq \omega$, and that $\diamondsuit^*_\kappa$ holds. Then* $(\kappa^+, \kappa) \not\twoheadrightarrow (\kappa, \leqq \rho)$.

PROOF OF PROPOSITION 32. See, for example, [1], theorem 3.2 and theorem 3.3 (where the principle is called $\diamondsuit''_\kappa$). Hence it is enough to show that, in M, $\diamondsuit^*_{\omega_3}$ holds. Let us set, for every regular cardinal ρ, $S_\rho = \{\alpha / \mathrm{cof}(\alpha) = \rho\}$. Applying [8], lemma 13, we see that $M \models \diamondsuit^*\ (\omega_3 \cap S_{\omega_2})$. On the other hand, applying [6], lemma 2.1, we see, since $M \models \mathrm{GCH}$, that $M \models$ "$\diamondsuit^*(\omega_3 \cap S_{\omega_1}) \wedge \diamondsuit^*(\omega_3 \cap S_{\omega_0})$". Hence $M \models \diamondsuit^*_{\omega_3}$, as was to be shown. QED Theorem 30

REMARK. Applying [8], lemma 13, again, we see that $M \models \diamondsuit^*_{\omega_1}$ as well. Hence $M \models (\omega_2, \omega_1) \not\twoheadrightarrow (\omega_1, \omega_0)$ (and there is no ω_2-saturated filter over ω_1, as well as no ω_4-saturated filter over ω_3).

REMARK. An immediate variation of the proof shows that the weak Conjecture of Chang for ω_3 (as defined in [4]) is also false in this model.

Let us end with an easy remark.

LEMMA 33. (1) *Assume that κ is κ^+-supercompact. Then* $\mathrm{CC}^+(\kappa^+, \kappa)$.

(2) *Assume that κ is $\kappa^{+\omega+1}$-supercompact. Then* $\mathrm{CC}^+(\kappa^{+\omega+1}, \kappa^{+\omega})$.

PROOF. (1) Take $j : V \rightarrow Q$ such that $j(\kappa) > \kappa^+$ and $Q^{\kappa^+} \subset Q$. Let $M = (\kappa^+, \kappa, A)$ be a structure. Then, $j \restriction_M : M \rightarrow j(M)$ lies in Q. So, in Q, $j(M)$ admits an elementary substructure $N = j''M$ such that $X \cap j\kappa = \kappa$ and $|X| = \kappa^+ = \kappa^{+O}$. Hence, in V, there exists a stationary set of α's such that M admits an elementary substructure $N = (X, X \cap \kappa, A \restriction_X)$, where $X \cap \kappa = \alpha$ and $|X| = \alpha^+$.

(2) Take $j : V \rightarrow Q$ such that $j(\kappa) > \kappa^{+\omega+1}$ and $Q^{\kappa^{+\omega+1}} \subset Q$. □

REMARK. In both cases, the theorem is still true with any language L such that $|L| < \kappa$. Moreover, we can assume that N is $<\alpha$-closed in M, i.e., that $M \cap N^{<\alpha} \subset N$.

REFERENCES

1. J. Baumgartner, *Ineffability properties of cardinals, II*, in *Logic, Foundations of Mathematics and Computability Theory* (Butts and Hintikka, eds.), D. Reidel Publ. Co., Dordrecht, Holland, pp. 87–106.

2. A. J. Dodd and R. B. Jensen, *The core model*, Ann. Math. Log. **20** (1981), 43–75.

3. H.-D. Donder, R. B. Jensen and B. Koppelberg, *Some applications of the core model*, in *Set Theory and Model Theory*, Springer Lecture Notes, No. 872 (1981), pp. 55–97.

4. H.-D. Donder and P. Koepke, *On the consistency strength of "accessible" Jonsson cardinals and of the weak Chang Conjecture*, Ann. Pure Appl. Log. **25** (1983), 233–261.

5. M. Foreman, *Large cardinals and strong model theoretic transfer properties*, Trans. Am. Math. Soc. **272** (1982), 427–463.

6. J. Gregory, *Higher Souslin trees and the generalised continuum hypothesis*, J. Symb. Log. **41** (1976), 663–671.

7. K. Kunen, *Saturated ideals*, J. Symb. Log. **43** (1978), 65–76.

8. J.-P. Levinski, *Filters and large cardinals*, Ann. Pure Appl. Log., to appear.

9. F. Rowbottom, *Some strong axioms of infinity incompatible with the axiom of constructibility*, Ann. Math. Log. **3** (1971), 1–44.

MATHEMATISCHES INSTITUT
FREIE UNIVERSITAET BERLIN
ARNIMALLEE 3, 1000 BERLIN 33, FRG